1.47 A computer chip manufacturer finds that, historically, for every 100 chips produced, 85 meet specifications, 10 need reworking, and 5 need to be discarded. Ten chips are chosen for inspection.

(a) What is the probability that all 10 meet specs?
(b) What is the probability that 2 or more need to be discarded?
(c) What is the probability that 8 meet specs, 1 needs reworking, and 1 will be discarded?

1.48 Unlike the city of Nirvana, New York, where 911 is the all-purpose telephone number for emergencies, in Moscow, Russia, you dial 01 for a fire emergency, 02 for the police, and 03 for an ambulance. It is estimated that emergency calls in Russia have the same frequency distribution as in Nirvana, namely, 60 percent are for the police, 25 percent are for ambulance service, and 15 percent are for the fire department. Assume that 10 calls are monitored and that none of the calls overlap in time and that the calls constitute independent trials.

1.49 A smuggler, trying to pass himself off as a glass-bead importer, attempts to smuggle diamonds by mixing diamond beads among glass beads in the proportion of one diamond bead per 1000 beads. A harried customs inspector examines a sample of 100 beads. What is the probability that the smuggler will be caught, that is, that there will be at least one diamond bead in the sample?

1.50 Assume that a faulty receiver produces audible clicks to the great annoyance of the listener. The average number of clicks per second depends on the receiver temperature and is given by $\lambda(\tau) = 1 - e^{-\tau/10}$, where τ is time from turn-on. Evaluate the formula for the probability of $0, 1, 2, \ldots$ clicks during the first 10 seconds of operation after turn-on. Assume the Poisson law.

1.51 A frequently held lottery sells 100 tickets at \$1 per ticket every time it is held. One of the tickets must be a winner. A player has \$50 to spend. To maximize the probability of winning at least one lottery, should he buy 50 tickets in one lottery or one ticket in 50 lotteries?

1.52 In the previous problem, which of the two strategies will lead to a greater expected gain for the player? The expected gain if $M (M \leq 50)$ lotteries are played is defined as $\overline{G}_M \triangleq \sum_{i=1}^{M} G_i P(i)$, where G_i is the gain obtained in winning i lotteries.

1.53 The switch network shown in Figure P1.53 represents a digital communication link. Switches α_i $i = 1, \ldots, 6$, are open or closed and operate independently. The probability that a switch is closed is p. Let A_i represent the event that switch i is closed.

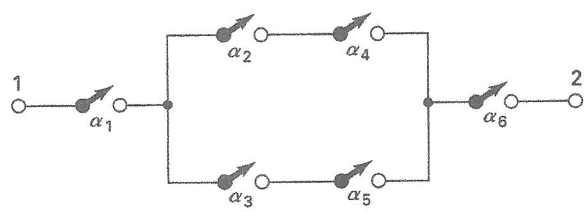

Figure P1.53 Switches in telephone link.

(a) In terms of the A_i's write the event that there exists at least one closed path from 1 to 2.
(b) Compute the probability of there being at least one closed path from 1 to 2.

1.54 (*independence of events in disjoint intervals for Poisson law*) The average number of cars arriving at a tollbooth per minute is λ and the probability of k cars in the interval $(0, T)$ minutes is

$$P(k; 0, T) = e^{-\lambda T}\frac{[\lambda T]^k}{k!}.$$

Consider two disjoint, that is, nonoverlapping, intervals, say $(0, t_1]$ and $(t_1, T]$. Then for the Poisson law:

$$P[n_1 \text{ cars in } (0, t_1] \text{ and } n_2 \text{ cars in } (t_1, T]] \quad (1.11\text{-}10)$$

$$= P[n_1 \text{ cars in } (0, t_1]]P[n_2 \text{ cars in } (t_1, T]], \quad (1.11\text{-}11)$$

that is events in disjoint intervals are independent. Using this fact, show the following:
(a) That $P[n_1 \text{ cars in } (0, t_1] | n_1 + n_2 \text{ cars in } (0, T]]$ is not a function of λ.
(b) In (a) let $T = 2$, $t_1 = 1$, $n_1 = 5$, and $n_2 = 5$. Compute $P[5 \text{ cars in } (0, 1] | 10 \text{ cars in } (0, 2]]$.

1.55 An automatic breathing apparatus (B) used in anesthesia fails with probability P_B. A failure means death to the patient unless a monitor system (M) detects the failure and alerts the physician. The monitor system fails with probability P_M. The failures of the system components are independent events. Professor X, an M.D. at Hevardi Medical School, argues that if $P_M > P_B$ installation of M is useless.† Show that Prof. X needs to take a course on probability theory by computing the probability of a patient dying with and without the monitor system in place. Take $P_M = 0.1 = 2P_B$.

1.56 In a particular communication network, the server broadcasts a packet of data (say, L bytes long) to N receivers. The server then waits to receive an acknowledgment message from each of the N receivers before proceeding to broadcast the next packet. If the server does not receive all the acknowledgments within a certain time period, it will rebroadcast (retransmit) the same packet. The server is then said to be in the "retransmission mode." It will continue retransmitting the packet until all N acknowledgments are received. Then it will proceed to broadcast the next packet. Let $p \triangleq P$[successful transmission of a single packet to a single receiver along with successful acknowledgment]. Assume that these events are independent for different receivers or separate transmission attempts. Due to random impairments in the transmission media and the variable condition of the receivers, we have that $p < 1$.

† A true story! The name of the medical school has been changed.

(a) In a fixed protocol or method of operation, we require that all N of the acknowledgments be received in response to a given transmission attempt for that packet transmission to be declared successful. Let the event $S(m)$ be defined as follows: $S(m) \triangleq$ {a successful transmission of one packet to all N receivers in m or fewer attempts}. Find the probability

$$P(m) \triangleq P[S(m)].$$

[*Hint*: Consider the complement of the event $S(m)$.]

(b) An improved system operates according to a dynamic protocol as follows. Here we relax the acknowledgment requirement on retransmission attempts, so as to only require acknowledgments from those receivers that have not yet been heard from on previous attempts to transmit the current packet. Let $S_D(m)$ be the same event as in part (a) but using the dynamic protocol. Find the probability

$$P_D(m) \triangleq P[S_D(m)].$$

[*Hint*: First consider the probability of the event $S_D(m)$ for an individual receiver, and then generalize to the N receivers.]

Note: If you try $p = 0.9$ and $N = 5$ you should find that $P(2) < P_D(2)$.

1.57 Toss two unbiased dice (each with six faces: 1 to 6), and write down the sum of the two face numbers. Repeat this procedure 100 times. What is the probability of getting 10 readings of value 7? What is the Poisson approximation for computing this probability? (*Hint*: Consider the event $A = \{sum = 7\}$ on a single toss and let p in Equation 1.9-1 be $P[A]$.)

1.58 On behalf of your tenants you have to provide a laundry facility. Your choices are

1. lease two inexpensive "Clogger" machines at \$50.00/month each; or
2. lease a single "NeverFail" at \$100/month.

The Clogger is out of commission 40 percent of the time while the NeverFail is out of commission only 20 percent of the time.

(a) From the tenant's point, which is the better alternative?
(b) From your point of view as landlord, which is the better alternative?

1.59 In the politically unstable country of Eastern Borduria, it is not uncommon to find a bomb onboard passenger aircraft. The probability that on any given flight, a bomb will be onboard is 10^{-2}. A nervous passenger always flies with an unarmed bomb in his suitcase, reasoning that the probability of there being *two bombs onboard* is 10^{-4}. By this maneuver, the nervous passenger believes that he has greatly reduced the airplane's chances of being blown up. Do you agree with his reasoning? If not, why not?

1.60 In a ring network consisting of eight links as shown in Figure P1.60, there are two paths connecting any two terminals. Assume that links fail independently with probability q, $0 < q < 1$. Find the probability of successful transmission of a packet from terminal A to terminal B. (Note: Terminal A transmits the packet in *both* directions on the ring. Also, terminal B removes the packet from the ring upon reception. Successful transmission means that terminal B received the packet from either direction.)

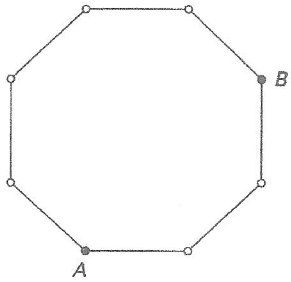

Figure P1.60 A ring network with eight stations.

1.61 A union directive to the executives of the telephone company demands that telephone operators receive overtime payment if they handle more than 5760 calls in an eight-hour day. What is the probability that Curtis, a unionized telephone operator, will collect overtime on a particular day where the occurrence of calls during the eight-hour day follows the Poisson law with rate parameter $\lambda = 720$ calls/hour?

1.62 Toss two unbiased coins (each with two sides: numbered 1 and 2), and write down the sum of the two side numbers. Repeat this procedure 80 times. What is the probability of getting 10 readings of value 2? What is the Poisson approximation for computing this probability?

1.63 The average number of cars arriving at a tollbooth is λ cars per minute and the probability of cars arriving is assumed to follow the Poisson law. Given that five cars arrive in the first two minutes, what is the probability of 10 cars arriving in the first four minutes?

***1.64** An aging professor, desperate to finally get a good review for his course on probability, hands out chocolates to his students. The professor's short-term memory is so bad that he can't remember which students have already received a chocolate. Assume that, for all intents and purposes, the chocolates are distributed randomly. There are 10 students and 15 chocolates. What is the probability that each student received at least one chocolate?

1.65 Assume that code errors in a computer program occur as follows: A line of code contains errors with probability $p = 0.001$ and is error free with probability $q = 0.999$. Also errors in different lines occur independently. In a 1000-line program, what is the approximate probability of finding 2 or more erroneous lines?

1.66 Let us assume that two people have their birthdays on the same day if both the month and the day are the same for each (not necessarily the year). How many people would you need to have in a room before the probability is $\frac{1}{2}$ or greater that at least two people have their birthdays on the same day?

1.67 (*sampling*) We draw ten chips at random from a semiconductor manufacturing line that is known to have a defect rate of 2 percent. Find the probability that more than one of the chips in our sample is defective.

***1.68** (*percolating fractals*) Consider a square lattice with N^2 cells, that is, N cells per side. Write a program that does the following: With probability p you put an electrically conducting element in a cell and with probability $q = 1-p$, you leave the cell empty. Do this for every cell in the lattice. When you are done, does there exist a continuous path for current to flow from the bottom of the lattice to the top? If yes, the lattice is said to *percolate*. Percolation models are used in the study of epidemics, spread of forest fires, and *ad hoc* networks, etc. The lattice is called a *random fractal* because of certain invariant properties that it possesses. Try $N = 10, 20, 50$; $p = 0.1, 0.3, 0.6$. You will need a random number generator. MATLAB has the function *rand*, which generates uniformly distributed random numbers x_i in the interval $(0.0, 1.0)$. If the number $x_i \leq p$, make the cell electrically conducting; otherwise leave it alone. Repeat the procedure as often as time permits in order to estimate the probability of percolation for different p's. A nonpercolating lattice is shown in Figure P1.68(a); a percolating lattice is shown in (b). For more discussion of this problem, see M. Schroeder, *Fractals, Chaos, Power Laws* (New York: W.H. Freeman, 1991).

***1.69** You are a contestant on a TV game show. There are three identical closed doors leading to three rooms. Two of the rooms contain nothing, but the third contains a $100,000 Rexus luxury automobile which is yours if you pick the right door. You are asked to pick a door by the master of ceremonies (MC) who knows which room contains the Rexus. After you pick a door, the MC opens a door (not the one you picked) to show a room *not* containing the Rexus. Show that even without any further knowledge, you will greatly increase your chances of winning the Rexus if you *switch* your choice from the door you originally picked to the one remaining closed door.

1.70 Often we are faced with determining the *more likely* of two alternatives. In such a case we are given two probability measures for a single sample space and field of events, that is, $(\Omega, \mathcal{F}, P_1)$ and $(\Omega, \mathcal{F}, P_2)$, and we are asked to determine the probability of an observed event E in both cases. The *more likely* alternative is said to be the one which gives the higher probability of event E.
Consider that two coins are in a box; one is "fair" with $P_1[\{H\}] = 0.5$ and one is "biased" with $P_2[\{H\}] = p$. Without looking, we draw one coin from the box and then flip this single coin ten times. We only consider the repeated coin-flips as our experiment and so the sample space $\Omega = \{$ all ten-character strings of H and T$\}$. We observe the event $E = \{$a total of four H's and six T's$\}$.

(a) What are the two probabilities of the observed event E, that is, $P_1[E]$ and $P_2[E]$?

(b) Determine the *likelihood ratio* $L \triangleq P_1[E]/P_2[E]$ as a function of p. (When $L > 1$, we say that the fair coin is more likely. This test is called a *likelihood ratio test*.)

Oscar's car has a $3.8 \times 10^{-3}/8.8 \times 10^{-2}$ or 0.043 probability of suffering a breakdown in the next 30 days.

Incidentally, the probability that a newly purchased Itsibitsi will have at least one breakdown in ten years is 0.95.

SUMMARY

The material discussed in this chapter is central to the concept of the whole book. We began by defining a real random variable as a mapping from the sample space Ω to the real line R. We then introduced a point function $F_X(x)$ called the cumulative distribution function (CDF), which enabled us to compute the probabilities of events of the type $\{\zeta\colon \zeta \in \Omega, X(\zeta) \leq x\}$. The probability density function (pdf) and probability mass function (PMF) were derived from the CDF, and a number of useful and specific probability laws were discussed. We showed how, by using Dirac delta functions, we could develop a unified theory for both discrete and continuous random variables. We then discussed joint distributions, the Poisson transform, and its inverse and the application of these concepts to physical problems.

We discussed the important concept of conditional probability and illustrated its application in the area of conditional failure rates. The conditional failure, often high at the outset, constant during mid-life, and high at old age, is fundamental in determining the probability law of time-to-failure.

PROBLEMS

(*Starred problems are more advanced and may require more work and/or additional reading.)

2.1 The event of k successes in n tries regardless of the order is the binomial law $b(k, n; p)$. Let $n = 10$, $p = 0.3$. Define the RV X by

$$X(k) = \begin{cases} 1, & \text{for } 0 \leq k \leq 2, \\ 2, & \text{for } 2 < k \leq 5, \\ 3, & \text{for } 5 < k \leq 8, \\ 4, & \text{for } 8 < k \leq 10. \end{cases}$$

Compute the probabilities $P[X = j]$ for $j = 1, \ldots, 4$. Plot the CDF $F_X(x) = P[X \leq x]$ for all x.

***2.2** Consider the probability space $(\Omega, \mathcal{F}, P)$. Give an example, and substantiate it in a sentence or two, where all outcomes have probability zero. Hint: Think in terms of random variables.

2.3 In a restaurant known for its unusual service, the time X, in minutes, that a customer has to wait before he captures the attention of a waiter is specified by the following CDF:

2.8 Compute $F_X(k\sigma)$ for the Rayleigh pdf (Equation 2.4-15) for $k = 0, 1, 2, \ldots$.

2.9 Write the *probability density functions* (using delta functions) for the Bernoulli, binomial, and Poisson PMF's.

2.10 The pdf of a RV X is shown in Figure P2.10. The numbers in parentheses indicate area. (a) Compute the value of A; (b) sketch the CDF; (c) compute $P[2 \leq X < 3]$;

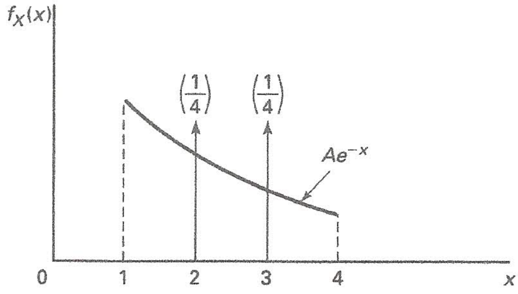

Figure P2.10 pdf of a Mixed RV.

(d) compute $P[2 < X \leq 3]$; (e) compute $F_X(3)$.

2.11 The CDF of a random variable X is given by $F_X(x) = (1 - e^{-x})u(x)$. Find the probability of the event $\{\zeta\colon X(\zeta) < 1 \text{ or } X(\zeta) > 2\}$.

2.12 The pdf of random variable X is shown in Figure P2.10. The numbers in parentheses indicate area. Compute the value of A. Compute $P[2 < X < 4]$.

2.13 (*two coins tossing*) The experiment consists of throwing two indistinguishable coins simultaneously. The sample space is Ω ={two heads, one head, no heads}, which we denote abstractly as $\Omega = \{\zeta_1, \zeta_2, \zeta_3\}$. Next, define two random variables as

$$X(\zeta_1) = 0,\ X(\zeta_2) = 0,\ X(\zeta_3) = 1$$
$$Y(\zeta_1) = 1,\ Y(\zeta_2) = -1,\ Y(\zeta_3) = 1.$$

(a) Compute all possible joint probabilities of the form $P[\zeta : X(\zeta) = \alpha, Y(\zeta) = \beta]$, $\alpha \in \{0, 1\}$, $\beta \in \{-1, 1\}$.

(b) Determine whether X and Y are independent random variables.

2.14 The pdf of the random variable X is shown in Figure P2.14. The numbers in parentheses indicate the corresponding impulse area.
So,

$$f_X(x) = \frac{1}{8}\delta(x+2) + \frac{1}{16}\delta(x+1) + \frac{1}{16}\delta(x-1) + \begin{cases} Ax^2, & |x| \leq 2, \\ 0, & \text{else.} \end{cases}$$

Note that the density f_X is zero off of $[-2, +2]$.

given that the normalized serving time of the teller x (i.e., the time it takes the teller to deal with a customer) is constant. However, the serving time is more accurately modeled as an RV X. For simplicity let X be a uniform RV with

$$f_X(x) = \frac{1}{5}[u(x) - u(x-5)].$$

Then $P[Y = k|X = x]$ is still Poisson but $P[Y = k]$ is something else. Compute $P[Y = k]$ for $k = 0, 1$, and 2. The answer for general k may be difficult.

2.20 Suppose in a presidential election each vote has equal probability $p = 0.5$ of being in favor of either of two candidates, candidate 1 and candidate 2. Assume all votes are independent. Suppose 8 votes are selected for inspection. Let X be the random variable that represents the number of favorable votes for candidate 1 in these 8 votes. Let A be the event that this number of favorable votes exceeds 4, that is, $A = \{X > 4\}$.

(a) What is the PMF for the random variable X ? Note that the PMF should be symmetric about $X = k = 4$.
(b) Find and plot the conditional distribution function $F_X(x|A)$ for the range $-1 \le x \le 10$.
(c) Find and plot the conditional pdf $f_X(x|A)$ for the range $-1 \le x \le 10$.
(d) Find the conditional probability that the number of favorable votes for candidate 1 is between 4 and 5 inclusive, that is, $P[4 \le X \le 5|A]$.

2.21 Random variables X and Y have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{4}x^2(1-y), 0 \le x \le 2, 0 \le y \le 1, \\ 0, \quad \text{else}. \end{cases}$$

Find:

(a) $P[X \le 0.5]$.
(b) $F_Y(0.5)$.
(c) $P[X \le 0.5|Y \le 0.5]$.
(d) $P[Y \le 0.5|X \le 0.5]$.

2.22 Consider the joint pdf of X and Y:

$$f_{XY}(x,y) = \frac{1}{3\pi}e^{-\frac{1}{2}[(x/3)^2+(y/2)^2]}u(x)u(y).$$

Are X and Y independent RVs? Compute the probability of $\{0 < X \le 3, 0 < Y \le 2\}$.

2.23 Consider the random variable X with pdf $f_X(x)$ given by

$$f_X(x) = \begin{cases} A(1+x), & -1 < x \le 0, \\ A(1-x), & 0 < x < 1, \\ 0, & \text{elsewhere}. \end{cases}$$

(a) Find A and plot $f_X(x)$;
(b) Plot $F_X(x)$, the pdf;
(c) Find point b such that

$$P[X > b] = \tfrac{1}{2}P[X \leq b].$$

2.24 Show that Equation 2.6-75 factors as $f_X(x)f_Y(y)$ when $\rho = 0$. What are $f_X(x)$ and $f_Y(y)$? For $\sigma = 1$ and $\rho = 0$, what is $P[-\frac{1}{2} < X \leq \frac{1}{2}, -\frac{1}{2} < Y \leq \frac{1}{2}]$?

***2.25** Consider a communication channel corrupted by noise. Let X be the value of the transmitted signal and Y be the value of the received signal. Assume that the conditional density of Y given $\{X = x\}$ is Gaussian, that is,

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(y-x)^2}{2\sigma^2}\right),$$

and X is uniformly distributed on $[-1, 1]$. What is the conditional pdf of X given Y, that is, $f_{X|Y}(x|y)$?

2.26 Consider a communication channel corrupted by noise. Let X be the value of the transmitted signal and Y the value of the received signal. Assume that the conditional density of Y given X is Gaussian, that is,

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-x)^2}{2\sigma^2}\right),$$

and that X takes on only the values $+1$ and -1 equally likely. What is the conditional density of X given Y, that is, $f_{X|Y}(x|y)$?

2.27 The arrival time of a professor to his office is a continuous RV uniformly distributed over the hour between 8 A.M. and 9 A.M. Define the events:

$$A = \{\text{The prof. has not arrived by 8.30 A.M.}\}, \tag{2.7-23}$$

$$B = \{\text{The prof. will arrive by 8:31 A.M.}\}. \tag{2.7-24}$$

Find

(a) $P[B|A]$.
(b) $P[A|B]$.

2.28 Let X be a random variable with *pdf*

$$f_X(x) = \begin{cases} 0, & x < 0, \\ ce^{-2x}, & x \geq 0, \end{cases} \qquad (c > 0).$$

(a) Find c;
(b) Let $a > 0$, $x > 0$, find $P[X \geq x + a]$;
(c) Let $a > 0$, $x > 0$, find $P[X \geq x + a | X \geq a]$.

2.29 To celebrate getting a passing grade in a course on probability, Wynette invites her Professor, Dr. Chance, to dinner at the famous French restaurant C'est Tres Chere. The probability of getting a reservation if you call y days in advance is given by $1 - e^{-y}$, where $y \geq 0$. What is the minimum numbers of days that Wynette should call in advance in order to have a probability of at least 0.95 of getting a reservation?